International Journal of Theoretical Physics, Vol. 32, No. 8, 1993

Quantum Mechanics of a Miniuniverse

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Received September 2, 1992

From the equations of general relativity for the radius of a closed homogeneous isotropic universe a Schrödinger equation for a particle is obtained. In the case of a universe filled with pressureless matter (dust) the equation is like that for the *s* states of a hydrogenlike atom. The miniuniverses obtained in this way have quantized masses of the order of the Planck mass.

1. INTRODUCTION

The purpose of this paper is to consider the expansion (or contraction) of a homogeneous isotropic universe regarded as quantum mechanical motion and described by the Schrödinger equation. One can regard this idea as associated in some sense with quantum gravity. However, the approach here is simpler and much more elementary than the usual works on quantum gravity (see, for example, Parker, 1984). The situation is somewhat analogous to the quantum mechanics of a single particle as compared to quantum field theory.

2. EQUATIONS OF GENERAL RELATIVITY

Let us consider a homogeneous isotropic universe described by the Robertson–Walker line element

$$ds^2 = c^2\,dt^2 - R^2\left(\frac{dr^2}{1-kr^2} + r^2\,d\theta^2 + r^2\sin^2\theta\,d\Phi^2\right) \tag{1}$$

where $R = R(t)$ and the curvature constant $k = 0, \pm 1$. If the universe is filled with matter (or radiation) characterized by a density $\rho(t)$ and a pressure $p(t)$, the Einstein field equations

$$G_\mu^\nu = -8\pi T_\mu^\nu \tag{2}$$

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0020-7748/93/0800-1435$07.00/0

lead to the relations (Weinberg, 1972, p. 472)

$$\frac{1}{c^2}\dot{R}^2 = \frac{8\pi}{3c^2}GR^2\rho - k \tag{3}$$

$$\frac{1}{c^2}\ddot{R} = -\frac{4\pi}{3c^2}GR\left(\rho + \frac{3}{c^2}p\right) \tag{4}$$

where conventional units are used and $\dot{R} = dR/dt$. For the consistency of these equations ρ and p must satisfy the condition

$$\dot{\rho} + \frac{3\dot{R}}{R}\left(\rho + \frac{p}{c^2}\right) = 0 \tag{5}$$

or

$$\frac{d\rho}{dR} = -\frac{3}{R}\left(\rho + \frac{p}{c^2}\right) \tag{6}$$

For a given equation of state relating ρ and p, (6) can be integrated to give $\rho = \rho(R)$. For example, in the case of dust with $p = 0$ one gets

$$\rho = \frac{A}{R^3} \qquad (A = \text{const}) \tag{7}$$

Let us multiply (3) by $\frac{1}{2}mc^2$, where m is the mass of the universe. We then have

$$\frac{1}{2}m\dot{R}^2 - \frac{4\pi}{3}mGR^2\rho = -\frac{1}{2}kmc^2 \tag{8}$$

This looks like the energy equation for a particle in one-dimensional motion,

$$T + V = E \tag{9}$$

where the particle coordinate is R, the kinetic energy is

$$T = \tfrac{1}{2}m\dot{R}^2 \tag{10}$$

the potential energy is

$$V(R) = -\frac{4\pi}{3}mGR^2\rho \tag{11}$$

and the total energy is

$$E = -\tfrac{1}{2}kmc^2 \tag{12}$$

If we multiply (4) by mc^2, we get the equation of motion of this particle,

$$m\ddot{R} = -\frac{dV}{dR} \tag{13}$$

with

$$\frac{dV}{dR} = \frac{4\pi}{3} mGR\left(\rho + \frac{3p}{c^2}\right) \tag{14}$$

One readily verifies that (14) follows from (11) if one makes use of (6).

The momentum of this particle is

$$P = m\dot{R} \tag{15}$$

and the Hamiltonian is

$$H = P^2/2m + V \tag{16}$$

It should be remarked that if, as above, one takes the mass of the universe m as a finite quantity, the universe must be closed, i.e., we must have

$$k = +1 \tag{17}$$

This will therefore be assumed. In that case we see from (12) that the energy $E < 0$. This is connected with the well-known fact that a closed universe expands to a maximum size and then contracts.

With the universe closed, one can consider R as its radius, and one readily finds with the help of (1) that its volume is given by

$$\tau = 2\pi^2 R^3 \tag{18}$$

3. SCHRÖDINGER EQUATION

So far we have been describing the particle motion classically. Let us now describe it in the framework of quantum mechanics. We define a wavefunction $\Psi = \Psi(R, t)$. Corresponding to (16), we take it to satisfy the usual Schrödinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial R^2} + V\Psi \tag{19}$$

For a stationary state with energy E we can write

$$\Psi = \Psi(R)\, e^{-(i/\hbar)Et} \tag{20}$$

and (19) becomes

$$-\frac{\hbar^2}{2m}\psi'' + V\psi = E\psi \tag{21}$$

with $\psi'' = d^2\psi/dR^2$.

4. A MODEL OF THE UNIVERSE

Let us take a simple model of a universe, one filled with pressureless matter, so that the density is given by (7). With the volume of this universe given by (18) the mass of the matter m_0 is given by

$$m_0 = \rho\tau = 2\pi^2 A \tag{22}$$

The total mass of the universe is m. According to (12) with $k = +1$, the mass associated with the motion is $-\frac{1}{2}m$. Hence we must have

$$m_0 = \tfrac{3}{2}m \tag{23}$$

so that

$$A = 3m/4\pi^2 \tag{24}$$

Then (11) gives

$$V = -Gm^2/\pi R \tag{25}$$

If in (21) we let

$$\psi = R\varphi \tag{26}$$

it can be written

$$-\frac{\hbar^2}{2m}\left(\varphi'' + \frac{2}{R}\varphi'\right) + V\varphi = E\varphi \tag{27}$$

With V given by (25) this looks like the Schrödinger equation in polar coordinates for the s states ($l = 0$) of a hydrogenlike atom (Messiah, 1961, p. 412) in which e^2 is replaced by Gm^2/π, i.e., we have what might be called a gravitational atom. For the bound states ($E<0$) the energy levels are given by

$$E_n = -\frac{G^2 m^5}{2\pi^2 \hbar^2 n^2} \qquad (n = 1, 2, 3, \ldots) \tag{28}$$

and the wavefunctions can be written in terms of Laguerre polynomials. Equating (28) to (12) with $k = +1$, we get for the mass m

$$m_n = (\pi \hbar c n/G)^{1/2} \tag{29}$$

so that the mass is different for different states.

For the ground state ($n = 1$) the mass is

$$m_1 = \pi^{1/2} m_p \tag{30}$$

where m_p is the Planck mass

$$m_p = (\hbar c/G)^{1/2} = 2.18 \times 10^{-5}\ \text{g} \tag{31}$$

The wavefunction is found to be

$$\psi_1 = 2a_1^{-3/2} R e^{-R/a_1} \tag{32}$$

where the "Bohr radius" a_1 is

$$a_1 = \pi \hbar^2 / G m_1^3 \tag{33}$$

and ψ_1 is normalized

$$\int_0^\infty \psi_1^2 \, dR = 1 \tag{34}$$

The size of the universe is of the order of

$$\bar{R}_1 = \int_0^\infty \psi_1^2 R \, dR = \tfrac{3}{2} a_1 \tag{35}$$

It should be noted that

$$a_1 = \pi^{-1/2} L_p \tag{36}$$

where L_p is the Planck length

$$L_p = (\hbar G/c^3)^{1/2} = 1.61 \times 10^{-33}\ \text{cm} \tag{37}$$

For an arbitrary value of n one has

$$m_n = (\pi n)^{1/2} m_p \tag{38}$$

and one finds that

$$\bar{R}_n = \tfrac{3}{2}\pi^{-1/2} n^{1/2} L_p \tag{39}$$

so that

$$\bar{R}_n = \frac{3}{2\pi} \frac{G}{c^2} m_n \tag{40}$$

It will be recalled that Gm_n/c^2 is the mass m_n in general relativity units.

If one goes over from mass to energy units, one finds that for the ground state

$$m_1 c^2 = 2.17 \times 10^{28} \text{ eV} \tag{41}$$

and for the excitation to the next state the energy required is

$$(m_2 - m_1)\, c^2 = 8.90 \times 10^{27} \text{ eV} \tag{42}$$

5. DISCUSSION

The large number of elementary particles (quarks and leptons) known at present suggests that they are not really elementary, but are made up of a smaller number of more fundamental particles, and it is plausible to suppose that the ultimate constituents have masses of the order of the Planck mass. Above it was found that in the case of a simple model of a homogeneous isotropic universe, if one describes its behavior by means of the Schrödinger equation, one gets miniuniverses with masses of this order of magnitude. However, these miniuniverses, regarded as elementary particles, suffer from two defects: they have no spin, and they are all neutral. The lack of spin is not a serious matter. If the center-of-mass motion of a particle is taken to be described by the Dirac equations, the particle acquires a spin. The problem of charge is more serious. To get a charged particle as a miniuniverse would require a much more complicated model of a universe than that considered above, if it is at all possible. However, the idea of an elementary particle being a quantized miniuniverse is very attractive and deserves further investigation.

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